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Boris S. Mordukhovich
Wayne State University, boris@math.wayne.edu

Lianwen Wang
Wayne State University

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BORIS MORDUKHOVICH and LIANWEN WANG

WAYNE STATE UNIVERSITY

Detroit, MI 48202

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Optimal Control of Differential-Algebraic Inclusions

Boris Mordukhovich[1] and Lianwen Wang[2]

[1] Department of Mathematics, Wayne State University boris@math.wayne.edu
[2] Department of Mathematics and Computer Science, Central Missouri State University

1 Introduction

In this paper we consider the following dynamic optimization problem (P) governed by *differential-algebraic inclusions*:

$$\text{minimize} \quad J[x,z] := \varphi(x(a),x(b)) + \int_a^b f(x(t),x(t-\Delta),\dot{z}(t),t)\,dt \tag{1.1}$$

subject to the constraints

$$\dot{z}(t) \in F(x(t),x(t-\Delta),t) \quad \text{a.e. } t \in [a,b], \tag{1.2}$$
$$z(t) = x(t) + Ax(t-\Delta), \quad t \in [a,b], \tag{1.3}$$
$$x(t) = c(t), \quad t \in [a-\Delta,a), \tag{1.4}$$
$$(x(a),x(b)) \in \Omega \subset I\!R^{2n}, \tag{1.5}$$

where $x : [a-\Delta,b] \to I\!R^n$ is continuous on $[a-\Delta,a)$ and $[a,b]$ (with a possible jump at $t=a$), and where $z(t)$ is absolutely continuous on $[a,b]$. We always assume that $F\colon I\!R^n \times I\!R^n \times [a,b] \rightrightarrows I\!R^n$ is a set-valued mapping of closed graph, that Ω is a closed set, that $\Delta > 0$ is a constant delay, and that A is a constant $n \times n$ matrix.

Differential-algebraic control systems are attractive mathematically (since they are essentially different from standard control systems even in the case of smooth dynamics) and very important for applications, especially in processes systems engineering. Necessary optimality conditions for controlled differential-algebraic *equations* with *no delays* are obtained in [11], where one can find detailed discussions and references on this topic. Let us mention also the research [1] on the so-called *implicit control systems* related to controlled differential-algebraic equations without delays. Necessary optimality conditions derived in these papers are based on reductions to standard (while nonsmooth) control systems by using uniform inverse mapping and implicit function theorems as well as on powerful techniques of nonsmooth analysis.

Note, however, that such reductions require rather restrictive assumptions of the "index one" type.

We are not familiar with any results in the literature on optimal control problems governed by differential-algebraic *inclusions* in either nondelayed or delayed settings. It seems that necessary optimality conditions for delayed differential-algebraic systems have not been specifically studied even in the case of controlled equations with smooth dynamics. On the other hand, differential-algebraic systems with delays are similar in many aspects to the so called *neutral functional-differential systems* that contain time-delays not only in state but also in velocity variables. Neutral systems have drawn much attention in the theory and applications of optimal control in the case of smooth dynamics. Necessary optimality conditions for nonsmooth neutral problems were first obtained in [9, 10] in the framework of neutral functional-differential inclusions. The techniques and constructions of [9, 10] are essentially used in what follows.

In this paper we derive necessary optimality conditions for the above problem (P) by *method of discrete approximations* developed in [7]. The results obtained are given in the forms of both Euler-Lagrange and Hamiltonian inclusions in terms of basic generalized differential constructions of variational analysis. We skip most of the proofs, which are similar to those given in [10] for the case of neutral systems.

2 Discrete approximations of differential-algebraic inclusions

This section deals with discrete approximations of an arbitrary admissible pair to the differential-algebraic system (1.2)–(1.4) without taking into account endpoint constraints. Let $(\bar{x}, \bar{z})$ be an admissible pair to (1.2)–(1.4), i.e., $\bar{x}(\cdot)$ is continuous on $[a-\Delta, a)$ and $[a, b]$ (with a possible jump at $t = a$) and $\bar{z}(\cdot)$ is absolutely continuous on $[a, b]$. The following assumptions are imposed throughout the paper:

(H1) There are an open set $U \subset I\!R^n$ and numbers $\ell_F, m_F > 0$ such that $\bar{x}(t) \in U$ for all $t \in [a-\Delta, b]$, the sets $F(x, y, t)$ are closed, and one has

$$F(x,y,t) \subset m_F I\!B,$$
$$F(x_1, y_1, t) \subset F(x_2, y_2, t) + \ell_F(|x_1 - x_2| + |y_1 - y_2|) I\!B$$

for all $(x, y, t) \in U \times U \times [a, b]$, $(x_1, y_1), (x_2, y_2) \in U \times U$, and $t \in [a, b]$, where $I\!B$ stands for the closed unit ball in $I\!R^n$.

(H2) $F(x, y, \cdot)$ is a.e. Hausdorff continuous on $t \in [a, b]$ uniformly in $U \times U$.

(H3) The function $c(\cdot)$ is continuous on $[a - \Delta, a]$.

Following [2], we consider the so-called *averaged modulus of continuity* for $F(x, y, t)$ with $(x, y) \in U \times U$ and $t \in [a, b]$ defined by

$$\tau(F;h) := \int_a^b \sigma(F;t,h)\,dt,$$

where $\sigma(F;t,h) := \sup\left\{\vartheta(F;x,y,t,h)\big|\ (x,y)\in U\times U\right\}$ with

$$\vartheta(F;x,y,t,h) :=$$
$$\sup\left\{\text{haus}(F(x,y,t_1),F(x,y,t_2))\ \Big|\ (t_1,t_2)\in\left[t-\tfrac{h}{2},t+\tfrac{h}{2}\right]\cap[a,b]\right\},$$

and where haus$(\cdot,\cdot)$ stands for the Hausdorff distance between two compact sets. It is proved in [2] that if $F(x,y,t)$ is Hausdorff continuous for a.e. $t\in[a,b]$ uniformly in $(x,y)\in U\times U$, then $\tau(F;h)\to 0$ as $h\to 0$.

Let us construct a sequence of discrete approximations of the given trajectory to the differential-algebraic inclusion replacing the derivative in (1.2) by the *Euler finite difference*

$$\dot z(t)\approx\frac{z(t+h)-z(t)}{h}.$$

For any $N\in I\!N:=\{1,2,\ldots\}$, we set $h_N:=\Delta/N$ and define the discrete mesh $t_j:=a+jh_N$ for $j=-N,\ldots,k$, and $t_{k+1}:=b$, where k is a natural number determined from $a+kh_N\le b<a+(k+1)h_N$. Then the corresponding *discrete systems* associated with (1.2)–(1.4) are given by

$$\begin{cases} z_N(t_{j+1})\in z_N(t_j)+h_NF(x_N(t_j),x_N(t_j-\Delta),t_j) & \text{for}\quad j=0,\ldots,k,\\ z_N(t_j)=x_N(t_j)+Ax_N(t_j-\Delta) \quad \text{for}\quad j=0,\ldots,k+1, \\ x_N(t_j)=c(t_j)\quad\text{for}\quad j=-N,\ldots,-1.\end{cases}\tag{2.1}$$

Given discrete functions $x_N(t_j)$ and $z_N(t_j)$ satisfying (2.1), we consider their *extensions* to the continuous-time intervals $[a-\Delta,b]$ and $[a,b]$, respectively, such that $x_N(t)$ are defined piecewise-linearly on $[a,b]$ and piecewise-constantly, continuously from the right on $[a-\Delta,a)$, while the "discrete velocities" $[z_N(t_{j+1})-z_N(t_j)]/h_N$ are extended to $[a,b]$ by

$$v_N(t):=\frac{z_N(t_{j+1})-z_N(t_j)}{h_N},\quad t\in[t_j,t_{j+1}),\ j=0,\ldots,k.$$

Let $W^{1,2}[a,b]$ be the standard Sobolev space with the norm

$$\|z(\cdot)\|_{W^{1,2}}:=\max_{t\in[a,b]}|z(t)|+\left(\int_a^b|\dot z(t)|^2dt\right)^{1/2}.$$

The following theorem establishes a *strong approximation* of any admissible trajectory for the differential-algebraic system by corresponding solutions to discrete approximations (2.1).

Theorem 2.1 *Let $(\bar x,\bar z)$ be an admissible pair for* (1.2)–(1.4) *under hypotheses (H1)–(H3). Then there is a sequence* $\{\Theta_N(t_j)\mid j=-N,\ldots,k+1\}$, $N\in$

$I\!N$, of solutions to discrete inclusions (2.1) such that $\Theta_N(t_0) = \bar{x}(a)$ for all $N \in I\!N$, the extensions $\Theta_N(t)$, $a - \Delta \le t \le b$, converge uniformly to $\bar{x}(\cdot)$ on $[a - \Delta, b]$ while $Z_N(t) := \Theta_N(t) + A\Theta_N(t - \Delta)$ converge to $\bar{z}(t) := \bar{x}(t) + A\bar{x}(t - \Delta)$ in the $W^{1,2}$-norm on $[a, b]$ as $N \to \infty$.

3 Strong convergence of discrete optimal solutions

This section constructs a sequence of *well-posed* discrete approximations for problem (P) such that *optimal solutions* to discrete approximation problems *strongly converge* to the *reference* optimal solution $(\bar{x}, \bar{z})$ of (P).

Given $\bar{x}(t)$, $a - \Delta \le t \le b$, take its approximation $\Theta_N(t)$ from Theorem 2.1 and denote $\eta_N := |\Theta_N(t_{k+1}) - \bar{x}(b)|$. Consider the following *discrete-time dynamic optimization problem* (P_N):

$$\begin{aligned} &\text{minimize} \quad J_N[x_N, z_N] := \varphi(x_N(t_0), x_N(t_{k+1})) + |x_N(t_0) - \bar{x}(a)|^2 \\ &+h_N \sum_{j=0}^{k} f(x_N(t_j), x_N(t_j - \Delta), \frac{z_N(t_{j+1}) - z_N(t_j)}{h_N}, t_j) \\ &+\sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| \frac{z_N(t_{j+1}) - z_N(t_j)}{h_N} - \dot{\bar{z}}(t) \right|^2 dt \end{aligned} \tag{3.1}$$

subject to the dynamic constraints (2.1), the *perturbed* endpoint constraints

$$(x_N(t_0), x_N(t_{k+1})) \in \Omega_N := \Omega + \eta_N I\!B, \tag{3.2}$$

and the auxiliary constraints

$$|x_N(t_j) - \bar{x}(t_j)| \le \varepsilon, \quad j = 1, \ldots, k+1, \tag{3.3}$$

with some $\varepsilon > 0$. The latter auxiliary constraints are needed to guarantee the existence of optimal solutions in (P_N) and can be ignored in the derivation of necessary optimality conditions.

In what follows we select $\varepsilon > 0$ in (3.3) such that $\bar{x}(t) + \varepsilon I\!B \subset U$ for all $t \in [a - \Delta, b]$ and take sufficiently large N ensuring that $\eta_N < \varepsilon$. Note that problems (P_N) have feasible solutions, since the pairs (Θ_N, Z_N) from Theorem 2.1 satisfy all the constraints (2.1), (3.2), and (3.3). Therefore, by the classical Weierstrass theorem in finite dimensions, each (P_N) admits an optimal solution $\bar{x}_N(\cdot)$ under the following assumption imposed in addition to the above hypotheses (H1)–(H3).

(H4) φ is continuous on $U \times U$, $f(x, y, v, \cdot)$ is continuous for a.e. $t \in [a, b]$ uniformly in $(x, y, v) \in U \times U \times m_F I\!B$, $f(\cdot, \cdot, \cdot, t)$ is continuous on $U \times U \times m_F I\!B$ uniformly in $t \in [a, b]$, and Ω is locally closed around $(\bar{x}(a), \bar{x}(b))$.

To justify the strong convergence of $(\bar{x}_N, \bar{z}_N) \to (\bar{x}, \bar{z})$ in the sense of Theorem 2.1, we need to involve an important intrinsic property of problem

(P) called *relaxation stability*. Consider the *relaxed problem* (R) of minimizing the cost functional (1.1) on admissible pairs to the *convexified* differential-algebraic system

$$\begin{cases} \dot{z}(t) \in \operatorname{co} F(x(t), x(t-\Delta), t) & \text{a.e. } t \in [a,b] \\ z(t) = x(t) + Ax(t-\Delta) & t \in [a,b] \end{cases} \tag{3.4}$$

subject to (1.4) and (1.5). Any admissible pair to (3.4) satisfying (1.4) is called a *relaxed pair* for (1.2)–(1.3).

One clearly has $\inf(R) \le \inf(P)$. The original problem (P) is said to be *stable with respect to relaxation* if $\inf(P) = \inf(R)$. This property, which obviously holds under the convexity assumption on the sets $F(x,y,t)$, goes far beyond the convexity. General sufficient conditions for the relaxation stability of problem (P) governed by differential-algebraic inclusions can be obtained similarly to those presented in [3] for the case of neutral inclusions. The next theorem makes a bridge between optimal control problems governed by differential-algebraic difference-algebraic control systems.

Theorem 3.1 *Let $(\bar{x}, \bar{z})$ be an optimal solution to problem (P), which is assumed to be stable with respect to relaxation. Suppose also that hypotheses (H1)–(H4) hold. Then any sequence $\{\bar{x}_N(\cdot), \bar{z}_N(\cdot)\}$, $N \in I\!N$, of optimal solutions to (P_N) extended to the continuous interval $[a-\Delta, b]$ strongly converges to $(\bar{x}, \bar{z})$ as $N \to \infty$ in the sense that $\bar{x}_N(\cdot)$ converge to $\bar{x}(\cdot)$ uniformly on $[a-\Delta, b]$ and $\bar{z}_N(\cdot)$ converge to $\bar{z}(\cdot)$ in the $W^{1,2}$-norm on $[a,b]$.*

4 Variational analysis in finite dimensions

To conduct a variational analysis of problems (P_N), which are intrinsically *nonsmooth*, we use appropriate tools of generalized differentiation introduced in [4] and then developed in many publications; see, e.g., the books [5, 13, 14, 15] for more details and references, where the notation $N(\cdot; \Omega)$, D^*F, and $\partial\varphi$ stand for the basic (limiting) normal cone to sets, the coderivatives of set-valued mappings, and the subdifferential of extended-real-valued functions.

The following two results are particularly important for our subsequent analysis. The first one obtained in [6] (see also [13, Theorem 9.40]), gives a complete coderivative characterization of the classical local Lipschitzian property of multifunctions.

Theorem 4.1 *Let $F\colon I\!R^n \rightrightarrows I\!R^m$ be a closed-graph multifunction locally bounded around $\bar{x}$. Then the following conditions are equivalent:*

(i) *F is locally Lipschitzian around $\bar{x}$.*
(ii) *There exist a neighborhood U of $\bar{x}$ and a number $\ell > 0$ such that*

$$\sup\Big\{|x^*| \ \Big|\ x^* \in D^*F(x,y)(y^*)\Big\} \le \ell|y^*| \quad x \in U,\ y \in F(x),\ y^* \in I\!R^m.$$

The next result (see, e.g., [5, Corollary 7.5] and [14, Theorem 3.17]) provides necessary optimality conditions for the following problem (MP) of *nonsmooth mathematical programming* with *many geometric constraints*, which is essential for the application to dynamic optimization:

$$\begin{cases} \text{minimize } \phi_0(w) \quad \text{subject to} \\ \phi_j(w) \le 0, \quad j = 1, \ldots, r, \\ g_j(w) = 0, \quad j = 0, \ldots, m, \\ w \in \Lambda_j, \quad j = 0, \ldots, l, \end{cases}$$

where $\phi_j : I\!\!R^d \to I\!\!R$, $g_j : I\!\!R^d \to I\!\!R^n$, and $\Lambda_j \subset I\!\!R^d$.

Theorem 4.2 *Let $\bar{w}$ be an optimal solution to (MP). Assume that all ϕ_i are Lipschitz continuous, that g_j are continuously differentiable, and that Λ_j are locally closed near $\bar{w}$. Then there exist real numbers $\{\mu_j | \, j = 0, \ldots, r\}$ as well as vectors $\{\psi_j \in I\!\!R^n | \, j = 0, \ldots, m\}$ and $\{w_j^* \in I\!\!R^d | \, j = 0, \ldots, l\}$, not all zero, satisfying the relations:*

$$\mu_j \ge 0 \quad for \quad j = 0, \ldots, r, \tag{4.1}$$

$$\mu_j \phi_j(\bar{w}) = 0 \quad for \quad j = 1, \ldots, r, \tag{4.2}$$

$$w_j^* \in N(\bar{w}; \Lambda_j) \quad for \quad j = 0, \ldots, l, \tag{4.3}$$

$$-\sum_{j=0}^{l} w_j^* \in \partial \Big(\sum_{j=0}^{r} \mu_j \phi_j \Big)(\bar{w}) + \sum_{j=0}^{m} \nabla g_j(\bar{w})^* \psi_j. \tag{4.4}$$

For formulating some results of this paper in the case of *nonautonomous* continuous-time systems we need certain extensions of the basic normal cone, subdifferential, and coderivative for the corresponding *moving* objects. These extensions denoted by $\widetilde{N}$, $\widetilde{\partial}$, and $\widetilde{D}^*$ reduce to the basic constructions under some natural assumptions; see [7, 8, 10] for more details and discussions.

5 Optimality conditions for difference-algebraic systems

In this section we derive necessary optimality conditions for the discrete approximation problems (P_N) by reducing them to those in Theorem 4.2 for nonsmooth mathematical programming. Given $N \in I\!\!N$, consider problem (MP) with the decision vector

$$w := (x_0^N, x_1^N, \ldots, x_{k+1}^N, v_0^N, v_1^N, \ldots, v_k^N) \in I\!\!R^{n(2k+3)}$$

and the following data:

$$\begin{aligned}
\phi_0(w) &:= \varphi(x_0^N, x_{k+1}^N) + |x_0^N - \bar{x}(a)|^2 + h_N \sum_{j=0}^{k} f(x_j^N, x_{j-N}^N, v_j^N, t_j) \\
&\quad + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| v_j^N - \dot{\bar{z}}(t) \right|^2 dt, \\
\Lambda_j &:= \left\{ (x_0^N, \ldots, v_k^N) \;\middle|\; v_j^N \in F(x_j^N, x_{j-N}^N, t_j) \right\}, \quad j = 0, \ldots, k, \\
\phi_j(w) &:= |x_j^N - \bar{x}(t_j)| - \varepsilon, \quad j = 1, \ldots, k+1, \\
\Lambda_{k+1} &:= \left\{ (x_0^N, \ldots, v_k^N) \;\middle|\; (x_0^N, x_{k+1}^N) \in \Omega_N \right\}, \\
g_j(w) &:= z_{j+1}^N - z_j^N - h_N v_j^N, \quad j = 0, \ldots, k,
\end{aligned}$$

where $z_j^N := x_j^N + A x_{j-N}^N$ for $j = 0, \ldots, k$, and where $x_j^N := c(t_j)$ for $j < 0$. Let $\bar{w}^N := (\bar{x}_0^N, \ldots, \bar{x}_{k+1}^N, \bar{v}_0^N, \ldots, \bar{v}_k^N)$ be an optimal solution to (MP). Applying Theorem 4.2, we find real numbers μ_j^N and vectors $w_j^* \in I\!R^{n(2k+3)}$ for $j = 0, \ldots, k+1$ as well as vectors $\psi_j^N \in I\!R^n$ for $j = 0, \ldots, k$, not all zero, such that conditions (4.1)–(4.4) are satisfied.

Taking $w_j^* = (x_{0,j}^*, \ldots, x_{k+1,j}^*, v_{0,j}^*, \ldots, v_{k,j}^*) \in N(\bar{w}^N; \Lambda_j)$ for $j = 0, \ldots, k$ and employing Theorem 3.1 on the convergence of discrete approximations, we have $\phi_j(\bar{w}^N) < 0$ for $j = 1, \ldots, k+1$ whenever N is sufficiently large. Thus $\mu_j^N = 0$ for these indexes due to the complementary slackness conditions (4.2). Let $\lambda^N := \mu_0^N \geq 0$. By the subdifferential sum rule for ϕ_0 defined above, inclusion (4.4) in Theorem 4.2 implies the relationships:

$$\begin{aligned}
-x_{0,0}^* - x_{0,N}^* - x_{0,k+1}^* &= \lambda^N u_0^N + \lambda^N h_N \vartheta_0^N + \lambda^N h_N \kappa_0^N \\
&\quad + 2\lambda^N (\bar{x}_0^N - \bar{x}(a)) - \psi_0^N + A^*(\psi_{N-1}^N - \psi_N^N), \\
-x_{j,j}^* - x_{j,j+N}^* &= \lambda^N h_N \vartheta_j^N + \lambda^N h_N \kappa_j^N + \psi_{j-1}^N - \psi_j^N \\
&\quad + A^*(\psi_{j+N-1}^N - \psi_{j+N}^N), \quad j = 1, \ldots, k-N, \\
-x_{k-N+1,k-N+1}^* &= \lambda^N h_N \vartheta_{k-N+1}^N + \psi_{k-N}^N - \psi_{k-N+1}^N - A^* \psi_k^N, \\
-x_{j,j}^* &= \lambda^N h_N \vartheta_j^N + \psi_{j-1}^N - \psi_j^N, \quad j = k-N+2, \ldots, k, \\
-x_{k+1,k+1}^* &= \lambda^N u_{k+1}^N + \psi_k^N, \\
-v_{j,j}^* &= \lambda^N h_N \iota_j^N + \lambda^N \theta_j^N - h_N \psi_j^N, \quad j = 0, \ldots, k
\end{aligned}$$

with the notation

$$(u_0^N, u_{k+1}^N) \in \partial \varphi(\bar{x}_0^N, \bar{x}_{k+1}^N), \quad (\vartheta_j^N, \kappa_{j-N}^N, \iota_j^N) \in \partial f(\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N, t_j),$$
$$\theta_j^N := 2 \int_{t_j}^{t_{j+1}} \left(\bar{v}_j^N - \dot{\bar{z}}(t) \right) dt.$$

Based on the above relationships, we arrive at the following result.

Theorem 5.1 *Let $(\bar{x}^N, \bar{z}_N)$ be an optimal solution to problem (P_N). Assume that the sets Ω and $\operatorname{gph} F_j$ are closed and that the functions φ and f_j are Lip-*

schitz continuous around the points $(\bar{x}_0^N, \bar{x}_{k+1}^N)$ *and* $(\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N)$, *respectively, for all* $j = 0, \ldots, k$. *Then there exist* $\lambda^N \geq 0$, p_j^N $(j = 0, \ldots, k+N+1)$ *and* q_j^N $(j = -N, \ldots, k+1)$, *not all zero, satisfying the conditions*

$$\begin{gathered}
p_j^N = 0, \quad j = k+2, \ldots, k+N+1, \\
q_j^N = 0, \quad j = k-N+1, \ldots, k+1, \\
(p_0^N + q_0^N, -p_{k+1}^N) \in \lambda^N \partial\varphi(\bar{x}_0^N, \bar{x}_{k+1}^N) + N((\bar{x}_0^N, \bar{x}_{k+1}^N); \Omega_N), \\
\Big(\frac{P_{j+1}^N - P_j^N}{h_N}, \frac{Q_{j-N+1}^N - Q_{j-N}^N}{h_N}, -\frac{\lambda^N \theta_j^N}{h_N} + p_{j+1}^N + q_{j+1}^N\Big) \\
\in \lambda^N \partial f_j(\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N) + N((\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{v}_j^N); \operatorname{gph} F_j), \quad j = 0, \ldots, k,
\end{gathered}$$

with the notation

$$\begin{gathered}
P_j^N := p_j^N + A^* p_{j+N}^N, \quad Q_j^N := q_j^N + A^* q_{j+N}^N, \\
\bar{v}_j^N := \frac{(\bar{x}_{j+1}^N + A\bar{x}_{j-N+1}^N) - (\bar{x}_j^N + A\bar{x}_{j-N}^N)}{h_N}.
\end{gathered}$$

6 Optimality conditions for differential-algebraic inclusions

Our main result establishes the following necessary optimality conditions of the *Euler-Lagrange* type for the original problem (P) derived by the limiting procedure from discrete approximations with the use of Theorem 4.1.

Theorem 6.1 *Let* $(\bar{x}, \bar{z})$ *be an optimal solution to problem* (P) *under hypotheses (H1)–(H4), where* φ *and* $f(\cdot,\cdot,\cdot,t)$ *are assumed to be Lipschitz continuous. Suppose also that* (P) *is stable with respect to relaxation. Then there exist a number* $\lambda \geq 0$ *and piecewise continuous functions* $p\colon [a, b+\Delta] \to I\!R^n$ *and* $q\colon [a-\Delta, b] \to I\!R^n$ *such that* $p(t) + A^*p(t+\Delta)$ *and* $q(t-\Delta) + A^*q(t)$ *are absolutely continuous on* $[a,b]$ *and the following conditions hold:*

$$\begin{gathered}
\lambda + |p(b)| = 1, \\
p(t) = 0 \quad t \in (b, b+\Delta], \\
q(t) = 0 \quad t \in (b-\Delta, b], \\
(p(a) + q(a), -p(b)) \in \lambda\partial\varphi(\bar{x}(a), \bar{x}(b)) + N((\bar{x}(a), \bar{x}(b)); \Omega), \\
\Big(\frac{d}{dt}[p(t) + A^*p(t+\Delta)], \frac{d}{dt}[q(t-\Delta) + A^*q(t)]\Big) \\
\in \operatorname{co}\Big\{(u,w) \mid (u, w, p(t)+q(t)) \in \lambda\widetilde{\partial} f(\bar{x}(t), \bar{x}(t-\Delta), \dot{\bar{z}}(t), t) \\
+\widetilde{N}\big((\bar{x}(t), \bar{x}(t-\Delta), \dot{\bar{z}}(t)); \operatorname{gph} F(\cdot,\cdot,t)\big)\Big\} \quad \text{a.e. } t \in [a,b].
\end{gathered}$$

Observe that for the Mayer problem (P_M), which is problem (P) with $f = 0$, the generalized Euler-Lagrange inclusions is equivalently expressed in terms of the extended coderivative with respect to the first two variables of $F = F(x, y, t)$, i.e., in the form

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t) + A^*p(t+\Delta)],\ \frac{d}{dt}[q(t-\Delta) + A^*q(t)]\Big)\\ &\in \operatorname{co} \widetilde{D}^*_{x,y} F\big(\bar{x}(t), \bar{x}(t-\Delta), \dot{\bar{z}}(t)\big)\big(-p(t) - q(t)\big) \quad \text{a.e. } t \in [a, b].\end{aligned}$$

It turns out that the extended Euler-Lagrange inclusion obtained above implies, under the relaxation stability of the original problems, two other principal optimality conditions expressed in terms of the Hamiltonian function built upon the mapping F in (1.2). The first condition called the *extended Hamiltonian inclusion* is given below in terms of a partial convexification of the basic subdifferential for the Hamiltonian function. The second one is a counterpart of the classical *Weierstrass-Pontryagin maximum condition* for hereditary differential-algebraic inclusions. Recall that a counterpart of the maximum principle does *not* generally hold for nonconvex differential-algebraic control systems even in the case of smooth dynamics.

The following relationships between the extended Euler-Lagrange inclusion and Hamiltonian inclusion are based on Rockafellar's dualization theorem [12] that concerns subgradients of abstract Lagrangian and Hamiltonian associated with set-valued mappings regardless the dynamics in. For simplicity we consider the case of the Mayer problem (P_M) for autonomous differential-algebraic inclusions. Then the Hamiltonian function for F is defined by

$$H(x, y, p) := \sup \big\{ \langle p, v \rangle \big|\ v \in F(x, y) \big\}.$$

Corollary 6.2 *Let $(\bar{x}, \bar{z})$ be an optimal solution to the Mayer problem (P_M) for the autonomous hereditary differential-algebraic inclusion* (1.2) *under the assumptions of Theorem 6.1. Then there exist a number $\lambda \geq 0$ and piecewise continuous functions $p\colon [a, b+\Delta] \to \mathbb{R}^n$ and $q\colon [a-\Delta, b] \to \mathbb{R}^n$ such that $p(t) + A^*p(t+\Delta)$ and $q(t-\Delta) + A^*q(t)$ are absolutely continuous on $[a, b]$ and, besides the necessary optimality conditions of Theorem 6.1, one has the extended Hamiltonian inclusion*

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t) + A^*p(t+\Delta)],\ \frac{d}{dt}[q(t-\Delta) + A^*q(t)]\Big)\\ &\in \operatorname{co} \Big\{ (u, w) \ \Big|\ \Big(-u, -w, \dot{\bar{z}}(t)\Big) \in \partial H\big(\bar{x}(t), \bar{x}(t-\Delta), p(t) + q(t)\big) \Big\}\end{aligned}$$

and the maximum condition

$$\big\langle p(t) + q(t), \dot{\bar{z}}(t) \big\rangle = H\big(\bar{x}(t), \bar{x}(t-\Delta), p(t) + q(t)\big)$$

for almost all $t \in [a, b]$. If moreover F is convex-valued around $(\bar{x}(t), \bar{x}(t-\Delta))$, then (6.1) *is equivalent to the Euler-Lagrange inclusion*

$$\begin{aligned}&\Big(\frac{d}{dt}[p(t)+A^*p(t+\Delta)],\ \frac{d}{dt}[q(t-\Delta)+A^*q(t)]\Big)\\&\in \operatorname{co} D^*F\big(\bar{x}(t),\bar{x}(t-\Delta),\dot{\bar{z}}(t)\big)\big(-p(t)-q(t)\big)\quad a.e.\ \ t\in[a,b],\end{aligned}$$

which automatically implies the maximum condition (6.1) *in this case.*

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